Viscous Flow in Multiparticle Systems: Motion of Spheres and a Fluid in a Cylindrical Tube

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An approximate theory for the behavior of multiparticle systems suspended in a viscous fluid is developed, based on a rigorous treatment for the case of a single sphere occupying any position in a cylindrical tube. The results obtained include estimates of the effect of some of the parameters involved on the particle velocities and spatial distribution of particles in very dilute sedimenting and fluidized beds as well as on the pressure drop resulting from passage of fluid. The conclusions presented are in agreement with such experimental data as are available and suggest a basis for more exact treatment of these systems.

The theoretical investigation reported here is a continuation of studies previously published (*8, 9*) having as their general objective a fundamental solution of the hydrodynamic relationships underlying low Reynolds-number phenomena involving particles suspended in a fluid. As in the case of the earlier studies the so-called *creeping-motion* equations are employed in conjunction with spherical-shaped particles in a cylindrical tube to furnish the idealization necessary for mathematical analysis. This limits the validity of the results to low Reynolds numbers. In the following treatment the simplest possible approximation is explored as a means of relating the phenomena of sedimentation, fluidization, and pneumatic conveying. Thus it will be expected to apply only in very dilute systems, but it is here (*8*) that the necessary boundary surface which laterally confines any actual bed is of greatest importance.

Attention is confined to the purely hydrodynamic aspects of the behavior of the particulate systems involved. Accordingly, approximations are presented for the effect of the various parameters involved on the particle velocities and spatial distribution of particles as well as the pressure drop experienced by passage of fluid. For any given system these parameters include the physical dimensions of spheres and cylinder, the specific gravity of the spheres, the viscosity and specific gravity of the fluid, fluid velocity, and geometrical distribution of particles entering the system. Only hydrodynamic forces are considered though in some cases interparticle friction and electrostatic effects (*10, 14*) may assume great importance. The treatment here should furnish a framework for further studies on systems possibly including heat, mass transfer, and chemical-reaction effects.

SINGLE-SPHERE BEHAVIOR

The procedure adopted here is first to develop the behavior of a single sphere in a cylinder and then to extend the treatment to more complicated cases involving more than one sphere. In order to furnish a suitable basis the case must be treated for a sphere which is free to occupy any position in a tube. Previously (*8*) the behavior of a single sphere suspended at the axis of a cylindrical tube was treated. Figure 1 illustrates the more general case now to be considered. The sphere moves with an arbitrary constant velocity U relative to the cylinder wall in the direction of Z positive, parallel to the axis of the cylinder, while the fluid is in laminar flow with a velocity U_{OF} (with respect to the cylinder wall) at the axis of the cylinder (average or superficial velocity $\frac{1}{2}U_{OF}$) at a sufficiently great distance from the sphere where the pattern is parabolic. The sphere radius is a, the cylinder radius is R_0, and the center of the sphere is located at a distance b from the cylinder axis. Cylindrical coordinates (R, Φ, Z) are employed that have their origin at the cylinder axis at the same elevation as the sphere center.

The equations of motion to be satisfied are, in vector notation,

$$\nabla^2 \mathbf{v} = \frac{1}{\mu} \nabla p \qquad (1)$$

together with the continuity equation for incompressible fluids,

$$\nabla \cdot \mathbf{v} = 0 \qquad (2)$$

where $\mathbf{v}$ is the fluid velocity with respect to an origin which moves with the sphere. The boundary conditions which define the field $\mathbf{v}$ are

$$\mathbf{v} = 0 \quad \text{at} \quad r = a \qquad (3)$$

$$\mathbf{v} = -\mathbf{i}_z U \quad \text{at} \quad R = R_0 \qquad (4)$$

At large distances from the sphere $z = \pm\infty$, the disturbance propagated by the sphere vanishes, and the fluid-velocity distribution becomes Poiseuillian. Hence, as an additional boundary condition,

$$\mathbf{v} = \mathbf{i}_z\left[U_{OF}\left(1 - \frac{R^2}{R_0^2}\right) - U\right] \qquad (5)$$

As in the previous study, this boundary-value problem is solved by the method of reflections, the solution consisting of the sum of a series of velocity fields all of which satisfy Equations (1) and (2) and each partially satisfies the boundary conditions as follows:

$$\mathbf{v}_0 = \mathbf{i}_z\left[U_{OF}\left(1 - \frac{R^2}{R_0^2}\right) - U\right] \qquad (6)$$

$$\mathbf{v}_1 = \begin{cases} -\mathbf{v}_0 \text{ at } r = a \\ 0 \text{ at } z = \pm\infty \text{ (i.e. } r = \infty) \end{cases} \qquad (7)$$

$$\mathbf{v}_2 = \begin{cases} -\mathbf{v}_1 \text{ at } R = R_0 \\ 0 \text{ at } Z = \pm\infty \end{cases} \qquad (8)$$

$$\mathbf{v}_3 = \begin{cases} -\mathbf{v}_2 \text{ at } r = a \\ 0 \text{ at } z = \pm\infty \text{ (i.e. } r = \infty), \end{cases} \qquad (9)$$

etc.

with as many such fields taken as needed for an appropriate degree of approximation. The field $\mathbf{v}$ satisfying the boundary conditions (3), (4), and (5) is then obtained in the form

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \cdots$$

and the corresponding pressure field,

$$p = p_0 + p_1 + p_2 + p_3 + \cdots.$$

The field $\mathbf{v}_1$, corresponding to a sphere suspended in a field which becomes parabolic at an infinite distance from it,

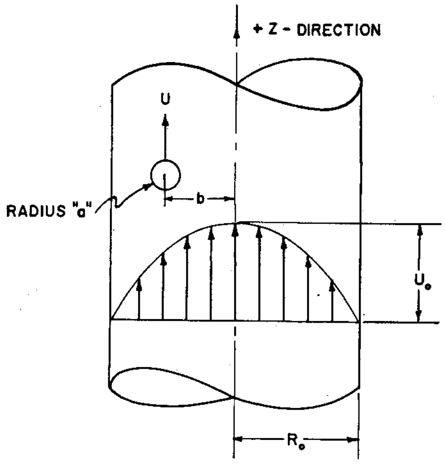

Fig. 1. Definition sketch—single sphere.

has been given by Simha (*18*) and Brenner (*1*). General expressions for additional reflections have been developed by Brenner and Happel (*2*), and details for the first two reflections are given in a supplement to this paper.*

For the present treatment only the first reflection is considered for the velocity field and the first two reflections are considered for pressure drop. In this approximation the effect of (a/R_0) is not evaluated and the "zeroth" approximation for drag and pressure drop is obtained as follows:

$$W = -6\pi\mu a \cdot \left[U - U_{OF}\left(1 - \frac{b^2}{R_0^2}\right)\right] \qquad (10)$$

$$\Delta P = \frac{12\mu a}{R_0^2}\left(1 - \frac{b^2}{R_0^2}\right) \cdot \left[(U_{OF} - U) - U_{OF}\frac{b^2}{R_0^2}\right] \qquad (11)$$

The expression for pressure drop corresponds to the energy dissipation experienced by a sphere moving in an unbounded medium with the same approach velocity.

MULTIPLE SPHERES

Relationships developed for a single sphere can be applied to the study of the behavior of more than one sphere by use of the same reflection technique. In order to illustrate this a simple case, involving the sedimentation of two spheres along the axis of a cylinder, is developed. The treatment, detailed in the supplement,* is confined to terms in first powers in a/R_0, and the investigation of Happel and Byrne (*8*) is used to obtain the velocity field generated by one of the spheres. More complicated cases can be developed by use of the same principle but substantial numerical calculations are involved.

For the case of multiple spheres the velocity contribution to the n-th sphere is obtained by summing the effects developed by each of the other $n - 1$ spheres. In the development which follows, the "zero" order approximation represented by Equations (10) and (11) is employed. Thus it is assumed that terms of the first power in a/R_0 and a/l may be omitted. (l is the distance between any two spheres.) The velocity field then will consist of the original undisturbed Poiseuille flow, except at the ends of the tube containing the spheres. The drag exerted on each particle will be given by Stokes's Law. Pressure drop due to each particle will depend solely upon its location and velocity.

Additional simplifications worth special attention result from the adoption of this simple hydrodynamic model. Since there is no interaction between particles, no prediction is possible of the effect of changes in the void volume between particles. It is assumed that the particles are sufficiently far apart so that such changes may be neglected. Actual bed depth corresponding to a given number of particles and fluid velocity must be specified independently. Similarly, on purely hydrodynamic grounds each particle will move only in an axial direction. No collisions are assumed and the pattern of motion continues unchanged along the entire tube length.

It is assumed that the particles suspended in a cylinder extend for an infinite distance axially. Boundary conditions at the inlet and outlet control the particle distribution which prevails in the tube. This particle distribution together with particle terminal settling velocity and fluid velocity constitute the three basic variables which influence the behavior of an assemblage. It is believed that a study of these idealized models will lead to a better understanding of the phenomena involved, though their quantitative application must be confined to assemblages of small particles located at relatively large distances from each other.

FIXED RELATIVE PARTICLE POSITION—UNIFORM DISTRIBUTION

First the simplest type of assemblage behavior will be considered, namely the case in which the particles suspended in a cylinder do not move relative to each other and in which they are randomly distributed throughout the cylinder cross section for an infinite distance axially. In practice this case is realized in sedimentation of a mass of particles in a quiescent fluid. The particles all fall at the same velocity and so maintain a fixed position relative to each other. This case also corresponds to the experiments of Happel and Epstein (*9*) in which pressure drop through rigid assemblages of particles was measured.

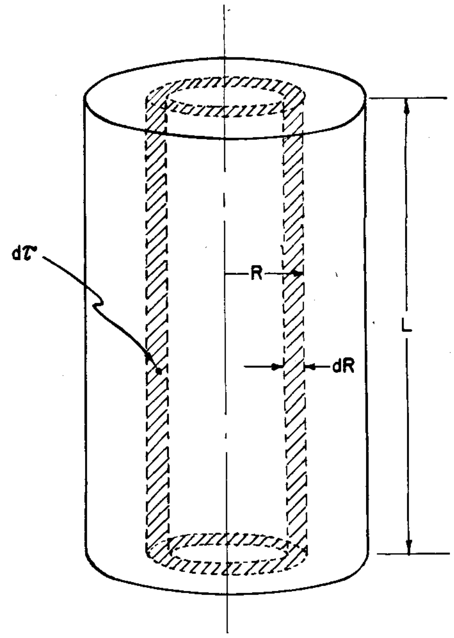

Fig. 2. Element of volume.

The number of particles N of radius a per unit volume is sufficient to characterize the behavior of such an assemblage. Since the particles do not influence each other by interaction fields, it is not necessary to know their exact arrangement as long as they are uniformly distributed.

A calculation will first be made for the velocity of such an assemblage by means of the annular volume $d\tau$ bounded by the radii R and $R - dR$ denoted in Figure 2.

The volume $d\tau = L(2\pi R\, dR)$, and the number of spheres contained in this volume is $NL(2\pi R\, dR)$. For this rigid assemblage the drag due to the presence of each sphere at position R is, Equation (10),

$$W = 6\pi\mu a\left[(U_{OF} - U) - U_{OF}\frac{R^2}{R_0^2}\right] \qquad (10a)$$

Hence the drag resulting from the particles in $d\tau$ is

$$dW_S = NL(2\pi R\, dR) \cdot 6\pi\mu a\left[(U_{OF} - U) - U_{OF}\frac{R^2}{R_0^2}\right] \qquad (12)$$

Integration gives the total drag,

$$W_S = NL(2\pi)(6\pi\mu a) \cdot \int_0^{R_0}\left[(U_{OF} - U)R - U_{OF}\frac{R^3}{R_0^2}\right] dR$$

whence

$$W_S = 6\pi\mu a \cdot N\pi R_0^2 L\left[\frac{U_{OF}}{2} - U\right] \qquad (13)$$

*Complete tabular material has been deposited as document 5441 with the American Documentation Institute, Photoduplication Service, Library of Congress, Washington 25, D. C., and may be obtained for $3.75 for photoprints or $2.00 for 35-mm. microfilm.

$U_{OF}/2$ is of course the average velocity of fluid in the cylinder and the result above is simply the equivalent of applying Stokes's Law to each of the spheres present. In terms of the terminal settling velocity of each of the particles it is readily shown by equating the drag to gravitational force that

$$U = U_{MF} - U_{TS}$$

where $U_{MF} = U_{OF}/2$, the mean fluid velocity.

A similar derivation for pressure drop is obtained by noting that the pressure drop due to a single sphere located at a distance R from the center is

$$\Delta P = \frac{12\mu a}{R_0^2}\left(1 - \frac{R^2}{R_0^2}\right) \cdot \left[(U_{OF} - U) - U_{OF}\frac{R^2}{R_0^2}\right] \qquad (11a)$$

Hence the pressure drop resulting from the particles in $d\tau$, with reference to Figure 3, is

$$d(\Delta P_S) = NL(2\pi R\, dR) \cdot \left(\frac{12\mu a}{R_0^2}\right)\left(1 - \frac{R^2}{R_0^2}\right) \cdot \left[(U_{OF} - U) - U_{OF}\frac{R^2}{R_0^2}\right] \qquad (14)$$

Integration between the limits of zero and R_0 gives the total pressure drop

$$\Delta P_S = NL8\pi\mu a\left(U_{MF} - \frac{3U}{4}\right) \qquad (15)$$

Here as in the remainder of this paper pressure drops due to the flow of fluid (Poiseuille's Law) and the effect of static head of the fluid are not included in the term ΔP_S, which refers only to the disturbance due to the presence of particles. For the case of sedimentation, where $U_{MF} = 0$, the pressure drop corresponds to the summation of the drag on the individual particles.

The case where particle velocity $U = 0$ gives the pressure drop through a rigid assemblage which does not move. If there are q particles contained in a volume of cylinder of length L and radius R_0, the following applies for the fractional void volume ϵ,

$$(1 - \epsilon) = \frac{\frac{4}{3}q\pi a^3}{\pi R_0^2 L} \qquad (16)$$

Thus the number of particles per unit volume N is

$$N = \frac{q}{\pi R_0^2 L} = \frac{3(1 - \epsilon)}{4\pi a^3} \qquad (17)$$

If this expression is substituted for N in Equation (15) and U is set equal to zero,

$$\Delta P_S = 6(1 - \epsilon)\frac{\mu U_{MF} L}{a^2} \qquad (18)$$

This expression differs from the exactly similar one derived by Happel and Byrne (*8*) and given by their Equation (64) by the constant, i.e., 9 instead of 6 as above. The assumption of Stokes's Law to compute the pressure drop would result in a factor of 4.5. Actual data on dilute assemblages by Happel and Epstein (*9*), as shown in their Figure 5 when extrapolated to infinite dilution, agree better with Equation (18) than either of the other values, providing confirmation for the present theoretical treatment.

MOBILE RELATIVE PARTICLE POSITION

Next the case will be considered in which the spheres are free to move relative to each other in an axial direction. In cases of practical interest radial distribution of particles may vary depending on end conditions, though distribution axially at any given radial location is assumed fixed. The general case for an arbitrary radial distribution will first be developed.

The frictional force, in the direction of flow, experienced by a sphere translating in the same direction with a constant velocity u when the fluid flows with a mean velocity $U_{MF} = \frac{1}{2}U_{OF}$ is given by Equation (10*a*). The gravitational force (corrected for the buoyancy of the fluid) experienced by the particle is

$$F_g = \frac{4\pi a^3(\rho_S - \rho_F)g}{3}$$

which, if Stokes's Law is valid for the terminal settling velocity in the quiescent fluid, is equivalent to

$$F_g = 6\pi\mu a U_{TS}$$

When no net force acts on the particle it will move with a constant velocity u, which is obtained by equating the drag and gravitational forces. The result obtained is

$$u = U_{OF}\left(1 - \frac{R^2}{R_0^2}\right) - U_{TS} \qquad (19)$$

which gives the equilibrium velocity of a particle situated at a distance R from the cylinder axis. Where all particles are free to move, a parabolic particle-velocity pattern thus results, as shown in Figure 3. Equation (19) implies that there is a radius s, $R_0 > s > 0$, for which the particle velocity $u = 0$. This radius might aptly be termed the *stagnation* radius and corresponds physically to the point at which the gravitational and frictional forces exactly balance one another. Upon putting $u = 0$ and $R = s$ in Equation (19) one obtains

$$\frac{s}{R_0} = \left(1 - \frac{U_{TS}}{U_{OF}}\right)^{1/2} \qquad (20)$$

In the region $R_0 \geq R > s$, which is hereafter termed the *outer annular area*, it is found from Equation (19) that $u < 0$, an indication that the particles in the vicinity of the tube wall have a downward motion. Similarly, in the *inner cylindrical space*, where $s > R \geq 0$, $u > 0$, corresponding to an upward motion of the solid particles in the vicinity of the cylinder axis. In the former case the net gravitational forces exceed those due to friction, as a result of the low fluid velocity in the neighborhood of the wall, accounting for the downward motion. The reverse effect predominates in the inner cylindrical space.

$$U_{OF}\left(1 - \frac{R^2}{R_0^2}\right) - u(R)$$

is the slip velocity and is constant with a value of U_{TS}, the terminal settling velocity.

The pressure drop through such an assemblage is for each sphere, situated at a distance R from the axis,

$$\Delta P_S = \frac{12\mu a}{R_0^2}\left(1 - \frac{R^2}{R_0^2}\right)U_{TS} \qquad (21)$$

Hence the pressure drop in the differential element of volume $d\tau$ is

$$d(\Delta P) = \frac{24\pi\mu a L}{R_0^2} \cdot U_{TS}\left(1 - \frac{R^2}{R_0^2}\right)nR\, dR \qquad (22)$$

Integration between the limits of zero and R_0 gives the over-all pressure drop. (L is assumed to be very large so that end effects may be neglected.) The number n of particles per unit volume at any radial location R is determined from end conditions. A general solution is possible without specific information regarding the dependence of n upon R, as follows. If ψ is the net number of particles transported per unit time in a positive axial direction, then

$$\int_0^{R_0} n \cdot u 2\pi R\, dR = \psi \qquad (23)$$

Whence,

$$2\pi\int_0^{R_0}\left[U_{OF}\left(1 - \frac{R^2}{R_0^2}\right) - U_{TS}\right]nR\, dR = \psi \qquad (24)$$

And finally

$$\int_0^{R_0}\left(1 - \frac{R^2}{R_0^2}\right)nR\, dR = \frac{\psi}{2\pi U_{OF}} + \frac{U_{TS}}{2\pi U_{OF}} \cdot \int_0^{R_0} n2\pi R\, dR \qquad (25)$$

However, if N_M is taken as the mean number of particles per unit volume averaged over the assemblage,

$$N_M \pi R_0^2 = \int_0^{R_0} n 2\pi R \, dR \qquad (26)$$

Thus the pressure drop is expressible as follows:

$$\Delta P = \frac{24\pi\mu a L}{R_0^2} U_{TS} \cdot \left[\frac{\psi}{2\pi U_{OF}} + \frac{U_{TS} N_M R_0^2}{2U_{OF}} \right] \qquad (27)$$

For the case of no net transport of particles, where total flow of particles upward balances downward flow, $\psi = 0$ and Equation (27) reduces to

$$\Delta P = N_M L 12\pi\mu a \frac{U_{TS}^2}{U_{OF}}. \qquad (28)$$

If desired, a formula for pressure drop corresponding to fractional void volume may be readily obtained by appropriate substitution for N_M from Equation (17),

$$\Delta P = \frac{9\mu L U_{TS}^2 (1 - \epsilon)}{2 U_{MF} a^2} \qquad (29)$$

Since the terminal settling velocity is given by

$$U_{TS} = \frac{2a^2 g}{9\mu} (\rho_S - \rho_F) \qquad (30)$$

and the weight of the bed of solid particles w, corrected for buoyancy, is given by

$$w = g(\rho_S - \rho_F) L (1 - \epsilon) \pi R_0^2 \qquad (31)$$

there results

$$\Delta P = \frac{U_{TS}}{U_{MF}} \times \frac{w}{\pi R_0^2} \qquad (32)$$

$\Delta P(\pi R_0^2)$ is the force F required to cause flow (above the Poiseuille Law pressure drop). Hence

$$F = w \times \frac{U_{TS}}{U_{MF}} \qquad (33)$$

Thus general expressions for velocity of particles and pressure drop have been developed. Next the effect of particle distribution will be considered. It should be noted that particle distribution will not affect particle velocity or pressure drop, except as it fixes the relationship between U_{TS} and U_{OF} required for specified particle transport ψ.

MOBILE BEDS

Uniform Particle Distribution

The simplest case for particle distribution in moving assemblages is that in which the radial particle distribution is uniform. One way in which this condition can be realized is by mixing particles and fluid uniformly at both ends and assuming that a steady state flow pattern is immediately established. The fluid entering a given zone in the tube will then contain a uniform particle concentration. This situation may be approximated in practice when both fluid and particles are introduced into the system simultaneously as in pneumatic conveying, pumping of slurries, and some sedimenting systems.

It is also likely that in fluidized systems where the velocity of fluid is close to the particle velocity, motion of particles at ends and other locations will cause fluid mixing so that a uniform particle distribution results.

In this case the variable concentration $n = n(R)$ is replaced by the constant concentration N. Equation (24) becomes

$$2\pi \int_0^{R_0} \left[U_{OF}\left(1 - \frac{R^2}{R_0^2}\right) - U_{TS} \right] N 2\pi R \, dR = \psi \qquad (34)$$

which is readily integrated. Upon setting $U_{OF}/2 = U_{MF}$, the simple result

$$U_{MF} - U_{TS} = \frac{\psi}{N\pi R_0^2} \qquad (35)$$

is obtained. For the case of no net particle transport, the "teeter" condition, $\psi = 0$ and

$$U_{MF} = U_{TS} \qquad (36)$$

Pressure drop through a uniform assemblage under appropriate restrictions upon ψ or U_{TS} may be determined by employing Equations (35) or (36) in conjunction with Equations (27) or (28) respectively. Thus for no net particle transport

$$\Delta P = NL6\pi\mu a U_{TS} = NL6\pi\mu a U_{MF} \qquad (37)$$

That is, the pressure drop is simply the equivalent of the Stokes's Law resistance corresponding to the terminal velocity of the particles.

Uniform Particle Flux

As noted above any radial distribution of particles is possible, depending on the conditions prevailing at the bed entrance and exit. It is desired to consider end conditions here which might be especially applicable in the case of fluidized beds, where the particles at entrance and exit conditions move independently of the fluid motion. In such beds the majority of particles are supported on a screen or other perforated device through which the fluid enters the bed. As circulation of particles occurs, the particles moving down are stopped abruptly at the screen and are redistributed. Again, fluid escapes from the top of the bed and the particles moving upward are redistributed as they assume a downward course. The present theory does not predict the depth of bed corresponding to a given number of particles because this necessitates a consideration of interaction effects

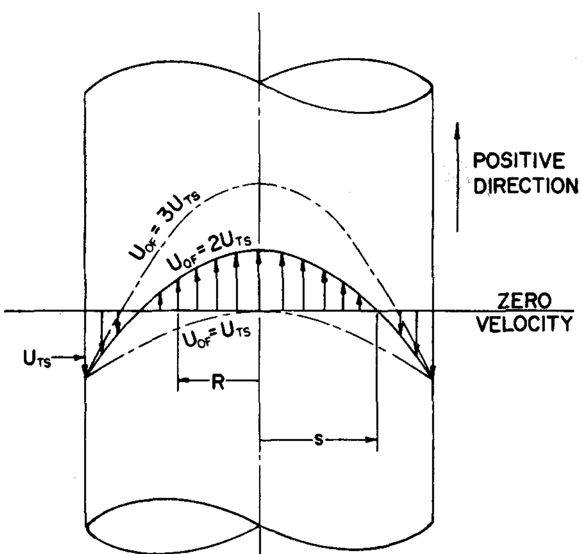

Fig. 3. Particle velocity pattern.

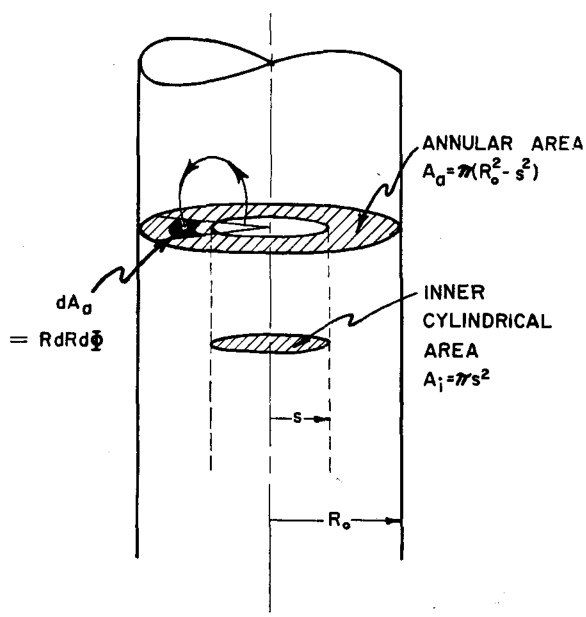

Fig. 4. Effect of end distribution.

among particles. However, the spatial distribution of particles, frequency of recirculation, and pressure drop can be computed on the presumption of random mixing of particles and fluid at the discontinuities appearing at the top and bottom of the bed.

In the development of the treatment to follow, it is assumed that a particle leaving the inner cylindrical space at the top of the bed has an equal probability of entering the outer annular space, down which particles are moving, at *any point.* The same reasoning applies for a particle leaving the annular space at the bottom of the tube and entering the inner cylindrical space. Other assumptions might be made, but it is believed that this assumption corresponds most closely to the facts in the present instance.

Conditions at the top of a bed are as shown in Figure 4. When a particle leaves the inner cylindrical space and enters the annular space, the probability that it enters the area dA_a if the particles are mixed is

$$\frac{dA_a}{A_a} = \frac{R\,dR\,d\Phi}{\pi(R_0^2 - s^2)} \qquad (38)$$

The relative distribution of particles to any differential area will be given by this fraction. If u is the particle velocity at any point, then the average particle velocity will be obtained by summing local velocities over the area and dividing by the total area through which they flow. In the case of a continuous distribution

$$U_{MS} = \frac{\int_A u\,dA}{A} \qquad (39)$$

Thus the average velocity of fall of a particle in the annular space will be

$$(U_{MS})_a = \frac{-\int_0^{2\pi}\int_{R=s}^{R=R_0} uR\,dR\,d\Phi}{\pi(R_0^2 - s^2)} \qquad (40)$$

$$= \frac{-2}{R_0^2 - s^2}\int_s^{R_0}\left[U_{OF}\cdot\left(1 - \frac{R^2}{R_0^2}\right) - U_{TS}\right]R\,dR \qquad (41)$$

$$= \frac{-2}{R_0^2 - s^2}\left[(U_{OF} - U_{TS})\cdot\frac{(R_0^2 - s^2)}{2} - \frac{U_{OF}}{R_0^2}\frac{(R_0^4 - s^4)}{4}\right] \qquad (42)$$

After further reduction,

$$(U_{MS})_a = -\left[\frac{U_{OF}}{2}\cdot\left(1 - \frac{s^2}{R_0^2}\right) - U_{TS}\right] \qquad (43)$$

whence, upon substitution of the value of s given in Equation (20),

$$(U_{MS})_a = \frac{U_{TS}}{2} \qquad (44)$$

A similar analysis is applicable to determine the velocity of particles in the inner cylindrical space

$$(U_{MS})_i = \frac{\int_0^{2\pi}\int_{R=0}^{R=s} uR\,dR\,d\Phi}{\pi s^2} \qquad (45)$$

$$= \frac{2}{s^2}\int_0^s\left[U_{OF}\cdot\left(1 - \frac{R^2}{R_0^2}\right) - U_{TS}\right]R\,dR \qquad (46)$$

$$= \frac{2}{s^2}\left[(U_{OF} - U_{TS})\cdot\frac{s^2}{2} - \frac{U_{OF}}{R_0^2}\frac{s^4}{4}\right] \qquad (47)$$

Consequently,

$$(U_{MS})_i = \frac{U_{OF} - U_{TS}}{2} \qquad (48)$$

The number of particles passing up through the entire inner cylindrical area per unit time, designated as ψ_i, will be a constant under steady state conditions. The number of particles passing down per unit time in the annulus will equal ψ_a. The number of particles passing a given area per unit time divided by their average velocity will give the number of particles contained in a unit length of path. The sum of these concentrations for upward- and downward-flowing particles must equal the average particle content of a unit length of bed. Thus

$$\frac{2\psi_i}{U_{TS}} + \frac{2\psi_a}{U_{OF} - U_{TS}} = N_M(\pi R_0^2) \qquad (49)$$

This provides a relation similar to Equation (35). The particle transport over the entire tube $\psi = \psi_i - \psi_a$. Thus in order to establish a relationship between U_{TS} and U_{OF}, ψ must be specified. To illustrate the procedure, the case for no net particle transport, $\psi = 0$, will be developed here. This results in $\psi_i = \psi_a$ expressed as follows:

$$\psi_i = \psi_a = \frac{N_M(\pi R_0^2)U_{TS}(U_{OF} - U_{TS})}{2U_{OF}} \qquad (50)$$

The inner cylindrical area may be expressed as

$$A_i = \pi s^2 = \frac{U_{OF} - U_{TS}}{U_{OF}}\pi R_0^2 \qquad (51)$$

Therefore the flux or number of particles

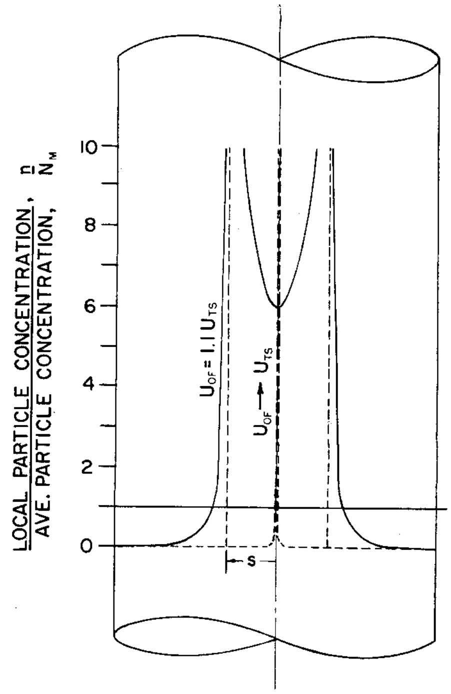

Fig. 5. Particle distribution—limit for upward transport.

per unit area per unit time passing through the inside area is

$$\frac{\psi_i}{A_i} = \frac{N_M U_{TS}}{2} \qquad (52)$$

This flux is constant over the cross-sectional area since equal distribution of particles was assumed over a given area of flow.

Similarly the area of the annular space is

$$A_a = \pi(R_0^2 - s^2) = \pi R_0^2\frac{U_{TS}}{U_{OF}} \qquad (53)$$

Therefore the flux through the annular space is

$$\frac{\psi_a}{A_a} = \frac{N_M(U_{OF} - U_{TS})}{2} \qquad (54)$$

The spatial distribution of particles may now be determined by noting that the flux at a given radial location is obtained by multiplying the local particle velocity by local concentration. Thus, for the annulus

$$\frac{\psi_a}{A_a} = \frac{N_M(U_{OF} - U_{TS})}{2} = n_a\left[U_{TS} - U_{OF}\left(1 - \frac{R^2}{R_0^2}\right)\right] \qquad (55)$$

whereupon

$$n_a = \frac{N_M(U_{OF} - U_{TS})}{2\left[U_{TS} - U_{OF}\left(1 - \frac{R^2}{R_0^2}\right)\right]}; \quad R_0 \geq R \geq s \qquad (56)$$

The number of particles per unit volume becomes very large (infinite) at the point $R = s$, where the particles are stationary.

Similarly the number of particles per unit volume in the inner cylindrical space n_i is

$$n_i = \frac{N_M U_{TS}}{2\left[U_{OF}\left(1 - \frac{R^2}{R_0^2}\right) - U_{TS}\right]}; \quad s \geq R \geq 0 \qquad (57)$$

Here again the number of particles per unit volume becomes very large at $R = s$.

Particle distribution in a system of this type will depend on the relative velocities U_{TS} and U_{OF}. It will be impossible to establish a condition of no net transport unless $U_{OF} = 2U_{MF} \geq U_{TS}$, because, as shown in Figure 3, no particle would have an upward velocity at lower fluid velocities. For this limiting case the particle concentration will be very high at the axis of the tube and fall off toward the walls, as illustrated in Figure 5. Cases have been reported in which this situation can be approached. Thus, Parent et al. (*17*) report "boiling," or aeration, at rates less than required to sweep even the finest particles of the contents out of a container.

The case where $U_{MF} = U_{TS}$ is of interest as this corresponds to the teeter condition for a bed of uniformly distributed particles. Pressure drop will be the same in both cases, though, as shown in Figure 6, particle distribution will be decidedly different. There is no theoretical upper limit to the value of U_{MF}, except that caused by difficulty in maintaining appropriate end conditions. Particle buildup at the walls becomes pronounced, as shown in Figure 6. This type of buildup of downward-moving particles in the vicinity of the walls has been repeatedly observed for fluidized beds (*4, 5, 11, 12, 15, 20*). In commercial fluidized beds velocities U_{MF} much higher than the terminal settling velocity have been observed (*6, 22*).

It is interesting to note that with the mechanism described here an increase in fluid velocity will result in a reduction in pressure drop for a given total bed weight supported, once $U_{TS} \geq U_{MF}$. This phenomenon was observed in experiments conducted by Lewis and Bowerman (*13*) in which cracking catalysts were fluidized at low velocities with liquid hydrocarbons. Pressure drops relative to bed weight of solids decreased 20% while the average fluid velocity increased threefold from the point of incipient fluidization. Further velocity increase by another tenfold factor resulted in a gradual approach of pressure drop to that required for $F = w$. Circulation can thus explain how particles without touching each other can cause pressure drop less than the weight of bed supported. In most commercial operations, where velocities are much higher than corresponds to incipient fluidization, the pressure drop is maintained approximately equal to the weight of suspended solids (*13, 14, 19*). Reduction in pressure drop with increase in velocity according to the mechanism treated here is an unstable phenomenon resulting from corresponding increase in particle segregation. Eventually a readjustment would occur owing to movement of particles back toward the middle of the tube, perhaps in an intermittent fashion as discussed by Miller and Logwinuk (*16*), with consequent increase in pressure drop. Of course, as more concentrated assemblages are employed, wall effects present in dilute suspensions assume a smaller relative importance in determining pressure drop, and so these analogies must be viewed with caution.

For the determination of pressure drop as a function of particle transport one may employ Equation (49) or (50) in conjunction with Equation (27) or (28) respectively. For the case of no net particle transport, $\psi = 0$, an expression for ΔP in terms of either U_{MF} or U_{OF} involves the additional parameter of particle circulation rate $\psi_i = \psi_a$. The greater the circulation rate, the larger will be the ratio of U_{MF} to U_{TS} required.

It is of some interest in connection with dilute fluidized beds to calculate the frequency of solids recirculation f for the case of no net transport $\psi = 0$. This quantity is defined as the number of complete cycles in the over-all circulation pattern which an average particle makes per unit time. In a bed of length L the average residence time of a particle falling in the annular space is, from Equation (44),

$$t_a = \frac{L}{U_{TS}/2} \qquad (58)$$

Similarly, the average residence time in the inner cylinder space, obtained from Equation (48), is

$$t_i = \frac{L}{(U_{OF} - U_{TS})/2} \qquad (59)$$

Thus, the average time required to complete one cycle is

$$t = t_a + t_i = \frac{2LU_{OF}}{U_{TS}(U_{OF} - U_{TS})} \qquad (60)$$

from which it follows that the frequency of recirculation is

$$f = \frac{1}{t} = \frac{U_{TS}(U_{OF} - U_{TS})}{2LU_{OF}} \qquad (61)$$

If θ is the nominal holdup time of the fluid as it traverses the bed,

$$\theta = \frac{L}{U_{MF}} = \frac{2L}{U_{OF}} \qquad (62)$$

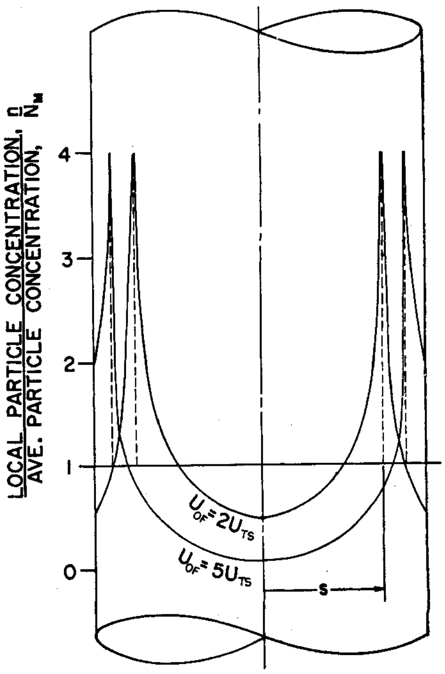

Fig. 6. Particle distribution—fluidized systems.

then the dimensionless parameter,

$$f\theta = \frac{U_{TS}}{U_{OF}}\left(1 - \frac{U_{TS}}{U_{OF}}\right) = \eta\left(\frac{U_{TS}}{U_{OF}}\right) \qquad (63)$$

may be of use in the correlation of backmixing effects.

DISCUSSION

The derivations above have been based on a simple set of consistent postulates in order to emphasize the basic conclusions. It is likely that certain other assumptions would provide better agreement with one or another set of empirical data. Thus one might choose to satisfy the continuity equation more exactly at higher solids concentrations by the assumption that instead of the original fluid velocity remaining undisturbed, it would be increased by a factor of $1/\epsilon$, (*3, 7*). This assumption in effect assumes complete shielding of each particle and is open to the objection that while it might approximate results at high concentrations of particles, its applicability would probably be questionable in dilute systems where wall effects might be important. Another more tenable assumption might involve removing the singularity noted in Equations (56) and (57) due to infinite particle concentration occurring at $R = s$. This could be done by noting that the concentration of particles could not occur to a greater extent than corresponds to fractional void volume of $\epsilon = 0.48$, the loose bed density (*9*). However, this would be inconsistent with the assumption of no particle interaction employed in developing the theory. Application of these semiempirical concepts in a sound way awaits further experimental validation.

The cases considered above were all concerned with the ideal case of steady flow in a vertical cylindrical tube of infinite length. In effect the system is assumed to be sufficiently dilute so that a wide range of particle concentrations might exist. It is hypothesized that some means must be provided at the ends of the tube to effect the particle distributions assumed for the cases of moving particles. Nothing in the theory developed explains how and where the top and bottom interfaces are produced in the case of a bed of finite length. In the laboratory screens could be provided at both bottom and top of a given section to accomplish this. In beds which are relatively short the effect of entrance and exit boundaries and fluid-flow patterns would also require consideration. Preliminary studies indicate that a possible way to take such effects into account would be to assume that the fluid flow to and from a bed would be produced by point sources and sinks rather than by an established parabolic flow pattern of fluid motion.

It should also be emphasized that the treatment applies only in very dilute systems and for the slow viscous flow region, that is essentially the range where Stokes's Law applies. Extension to higher concentrations could be accomplished by taking into account additional reflections as discussed in the supplement. To form a quantitative estimate of the range of concentration for which the present theory is applicable, it would be necessary to develop at least the first interaction between spheres and cylinder walls.

The assumption of a parabolic distribution of fluid is also consistent with the idealization of no interaction of spheres with each other or the cylinder walls. Another extreme of behavior of exceedingly dilute systems would involve the case where the particle surface once was extremely large compared with that of the walls, in which case the average fluid velocity would be uniform. To determine the combined effect it would again be necessary to develop the first interaction. For cases of higher velocity where inertial effects cannot be neglected, the linearized form of the Stokes-Navier equations exployed here wil̥l not be adequate. However, it should be noted that often the motion of particles relative to fluid will be very small even though rate of movement of particles and fluid with respect to container walls is substantial.

It is thought that, in spite of many qualifications, the theory is of interest in providing a start toward a unified picture of the phenomena of fluidization and sedimentation based on fundamental fluid dynamic considerations. This is considered worthwhile because many laboratory-scale investigations do not appear to take into account the numerous variables which exist in commercial applications of these processes.

For dilute beds where depth is great enough so that entrance and exit effects are not important, the present theory can be employed to predict other characteristics of interest in the application of moving-bed systems. Thus, given an appropriate bed depth corresponding to a definite concentration of particles, one may estimate such items as fraction of particles present in annular space and in internal cylindrical space and time of contact of fluid with particles in the bed.

CONCLUSIONS

It has been demonstrated that a simple hydrodynamic theory based on the motion of a dilute system of spherical particles in a cylinder through which fluid is slowly passing may throw some light on phenomena observed in practical applications involving behavior of particles suspended in fluids.

Thus recirculation effects and distribution of particles can be examined in a qualitative fashion. Pressure drops are shown to be greatest relative to bed weight supported in the case of fixed assemblages. Recirculation of a uniformly dispersed suspension results in a lowering of pressure drop. If redistribution as well as recirculation occurs, still lower pressure drops are attainable but an unstable condition results.

Further work in progress aims to continue these studies from both theoretical and experimental approaches. It is believed that this paper represents progress in understanding the microscopic effects involved, as contrasted with statistical or empirical approaches to the problem.

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NOTATION

(Consistent Absolute Units)

a = radius of spherical particle
A = area
b = distance of sphere center from cylinder axis
F = force required to maintain fluid flow due to presence of spheres
F_g = gravitational force (corrected for buoyancy) acting on a spherical particle
f = frequency of particle recirculation
g = acceleration of gravity
l = distance between two spheres
L = length of an assemblage of spheres
n = number of spheres per unit volume at a given point location
N = constant number of spheres per unit volume
N_M = mean number of spheres per unit volume averaged over entire assemblage
p = viscous pressure at any location
ΔP = pressure drop due to presence of sphere or spheres
q = number of particles in a cylinder of length L and radius R_0
R = perpendicular distance from longitudinal axis of cylinder
R_0 = radius of cylinder
s = distance from cylinder axis at which there is no relative motion of sphere with respect to cylinder wall
t = average residence time of a particle
u or U = velocity of sphere or spheres in $+Z$ direction with respect to cylinder wall
U_{OF}, U_{OS} = axial velocity of fluid or spheres, respectively, in direction of Z positive with respect to cylinder wall
U_{MF}, U_{MS} = mean velocity of fluid or spheres, respectively, in direction of Z positive with respect to cylinder wall
U_{TS} = terminal settling velocity in direction of Z negative with respect to cylinder wall
w = weight of assemblage of particles corrected for buoyancy
W = frictional force in $+Z$ direction

Greek Letters

ϵ = fractional void volume
η = function in Equation (62)
θ = nominal holdup time of fluid
μ = viscosity
π = constant, 3.14159 $\cdots$
ρ = density
τ = volume
ψ = particle flux, particles per unit time

Vector Quantities

∇ = gradient
$\nabla\cdot$ = divergence
∇^2 = Laplace operator
$\mathbf{i}$ = unit vector; subscript indicates direction of vector
$\mathbf{v}$ = fluid velocity at any point

Subscripts

0, 1, 2, $\cdots$ = different velocity fields
a = outer annular area, where particles are moving down
A = a sphere
B = a sphere
F = fluid
i = inner cylindrical space where particles are moving up
m = mean value

0 = axial position
R_0 = evaluation of the function at $R = R_0$
S = sphere

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Unsteady State Heat Transfer in Stationary Packed Beds

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A new solution is presented of the differential equations describing unsteady state heat transfer in stationary beds of small granular solid particles through which a fluid is flowing. Arbitrary initial solid temperature distribution and arbitrary variation of inlet gas temperature are allowed. The solution presented appears easier to apply in practice than those previously published and affords an example of the versatility of Fourier integrals and series. An application of the solution to the regeneration of Dow type-B butylene dehydrogenation catalyst is described.

Passage of a fluid through a bed of granular solid is of common occurrence in chemical engineering practice. Several mathematical treatments of unsteady state heat transfer in this situation have been published. Schumann (*1*) developed the basic differential equations and presented a solution for the case of uniform initial solid temperature and constant entering fluid temperature. His work was extended by Furnes (*2*), Goldstein (*3*), and others. More general cases have been covered by Amundson (*4, 5*). One (*4*) considers the unsteady state with arbitrary initial solid temperature and also arbitrary inlet fluid temperature at any time. This publication was presented as a solution to a problem in adsorption, but the mathematical statement is identical with that for the heat transfer problem. A more recent publication by Amundson (*5*) covers much more general cases where heat may be produced or absorbed in the bed, simultaneously transferred through the wells of the containing vessel, etc.

All solutions published to date have been in terms of Bessel functions or other functions which are published for limited ranges of variables or at best for relatively widely separated values. Also, in most practical cases the more general solutions are quite laborious to apply. Presented here are two solutions of the same problem as that treated by Amundson (*4*), but they are derived from Fourier integrals and Fourier series. The resulting mathematical forms are consequently easy to apply. Convergence is rapid in most cases. The series form of solution is an approximation but in most practical cases it is a very good one. In general, it is easier to use than the integral form, which however is exact.

The problem concerns a bed in the shape of a right porous prism (or cylinder) of granular material the initial temperature of which is an arbitrary function of distance into the bed. A fluid, the inlet temperature of which is an arbitrary function of time, is allowed to pass lengthwise through the bed at a uniform rate of flow, the sides of the bed being impervious to the fluid and to heat. The problem is to find the distribution of temperature in the solid material and in the fluid for all time if it is assumed that

1. The gas and solid temperature are uniform across any section perpendicular to the axis of the prism.
2. The solid particles are so small or have such high thermal diffusivity that any given particle may be considered as being at a uniform temperature at any instant.
3. Compared with the transfer of heat from fluid to solid, the transfer of heat by conduction, convection, or mixing in the fluid itself or in the solid itself is small and may be neglected.
4. The rate of heat transfer from fluid to solid at any point is proportional to the difference in temperature between fluid and solid at that point.
5. Change in volume of fluid and solid due to change in temperature may be neglected.
6. The thermal constants are independent of the temperature.
7. There is no generation or absorption of heat as latent heat, heat of chemical reaction, etc.

DERIVATION OF EQUATIONS

The basic differential equations describing the problem, as derived by Schumann (*1*), are

$$\frac{\partial t_g}{\partial \theta} + \frac{g}{\rho_g f}\frac{\partial t_g}{\partial x} = -\frac{hs}{C_g \rho_g f}(t_g - t_s) \quad (1)$$

$$\frac{\partial t_s}{\partial \theta} = \frac{hs}{\rho_s C_s}(t_g - t_s) \quad (2)$$

Changing the independent variables to y and z gives

$$\frac{\partial t_g}{\partial y} = t_s - t_g \quad (3)$$